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ROCKET BOOSTER CONTROL

PENALTY FUNCTIONS AND BOUNDED PHASE COORDINATE CONTROL

by D. L. Russell

Prepared under Contract No. NASw-563 *by*
MINNEAPOLIS-HONEYWELL REGULATOR COMPANY
Minneapolis, Minnesota
for

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PENALTY FUNCTIONS AND BOUNDED PHASE COORDINATE CONTROL*

By D. L. Russell‡

ABSTRACT

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The use of two different kinds of penalty functions to obtain approximate and, in the limit, exact solutions to the bounded phase coordinate optimal control problem is considered. The first type of penalty function assumes small values within the phase constraint and large values outside while the second type is defined only within the phase constraints, assuming small values away from the constraint boundary but increasing to infinity as that boundary is approached.

The bounded, or constrained phase problem is replaced by a new problem in which the constraints are formally removed but the cost functional is augmented by adding a penalty function of one of the above types. Solutions to this new problem are shown to exist. They are seen, under suitable hypotheses, to be approximations to the solutions of the original constrained problem in the sense that as the severity of the penalty imposed increases without limit these unconstrained solutions converge, in an appropriate sense, to solutions of the constrained problem.

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INTRODUCTION

Synthesis of controllers that satisfy an optimization criterion and which do not force the system response trajectories beyond prescribed boundaries in phase space is a difficult problem that has not been satisfactorily resolved. Although direct approaches to the problem have met with some success, it is proper to consider other methods.

Instead of trying to develop a synthesis procedure which determines an optimal trajectory that does not exceed the phase constraint, an unconstrained problem is considered wherein the original cost functional is augmented by a non-negative penalty function which sharply increases the cost associated with trajectories lying outside the phase constraints. By using sequences of cost functionals with more and more severe penalty functions, it might be expected that a desired constrained phase solution could be approximated to any desired degree of accuracy.

This document deals only with the existence of solutions and with their convergence properties. Necessity, sufficiency, and uniqueness considerations as well as computational methods are subjects for later investigations.

In the PROBLEM STATEMENT which follows, two different kinds of penalty functions are defined and the optimization problem is posed. OPTIMIZATION WITH A PENALTY FUNCTION establishes the existence of at least one controller that minimizes a cost functional whose integrand has been modified by addition of a penalty function. This result is then used in the existence and convergence result proved in OPTIMIZATION WITH SEQUENCES OF PENALTY FUNCTIONS. Results of this document are then summarized in the CONCLUSIONS.

PROBLEM STATEMENT

The optimization problem is posed and two types of penalty functions are defined.

<u>THE OPTIMIZATION PROBLEM</u> The system of differential equations

$$\dot{x}^i = g^i(t,x) + \sum_{j=1}^{m} h_j^i(t,x)u^j(t) \qquad (1')$$

or, in vector notation

$$\dot{x} = g(t,x) + H(t,x)\,u(t) \qquad (1)$$

is considered. Here x and u are n and m dimensional vectors, respectively, while H is an nxm matrix. $g^i(t,x)$ and $h_j^i(t,x)$ are continuous and have continuous first order partial derivatives with respect to x in I x G. The existence is assumed of

(a) a closed constraint set $G \subseteq E^n$ possessing a non-empty interior,

(b) a compact, convex subset $\Omega \subset E^m$ possessing at least two points,

(c) a compact subinterval I of E^1 with non-zero length

(d) a compact target subset T(t) of Int(G) varying continuously with $t \in I$, and a point $x_o \in$ Int(G) such that for $t \in I$, $x_o \notin T(t)$.

(e) a non-empty set Δ of measurable vector functions u defined on various intervals $[t_o,t_1] \subseteq I$ such that the unique solutions x(t) of (1) with $x(t_o) = x_o$ corresponding to these $u(t)$ obey

(i) $x(t_1) \in T(t_1)$,

(ii) $x(t) \in G$ for $t \in [t_o, t_1]$,

(f) a cost functional

$$C(u) = \int_{t_o}^{t_1} \{g^o(t,x) + \sum_{j=1}^{m} h_j^o(t,x)u^j(t)\}dt \tag{2}$$

defined for each u in Δ. Here g^o and h_j^o are continuous in $I \times G$.

PROBLEM Find $\bar{u} \in \Delta$ such that for all $u \in \Delta$, $C(\bar{u}) \leq C(u)$.

If in (a) the set G is taken to be E^n, the problem will be called unconstrained. Otherwise it will be said that the optimization problem is constrained.

A solution to this problem will be sought by use of sequences of penalty functions.

DEFINITION 1. The sequence of functions $\{p_i(x)\}$ is said to be a sequence of penalty functions of the first kind for G in case:

(a) There is an open set O with $G \subset O \subseteq E^n$ such that for each $i = 1,2,\ldots p_i(x)$ is defined and continuous on O and there assumes non-negative, real values.

(b) Given any compact set $D \subset \text{Int}(G)$, $\lim_{i \to \infty} p_i(x) = 0$ uniformly for $x \in D$.

(c) Given any compact set $D \subset \text{Ext}(G) \cap O$, $\lim_{i \to \infty} p_i(x) = +\infty$ uniformly for $x \in D$.

Usually the open set O will be taken to be E^n.

DEFINITION 2. The sequence of functions $\{p_i(x)\}$ is said to be a sequence of penalty functions of the second kind for G in case:

(a) For each $i = 1,2,\ldots p_i(x)$ is defined, continuous, and assumes real, non-negative values in Int(G).

(b) Given any compact set $D \subset \mathrm{Int}(G)$, $\lim_{i \to \infty} p_i(x) = 0$ uniformly for $x \in D$.

(c) Let ∂G denote the boundary of G. Then for every $i = 1,2,\ldots$ $\lim_{\mathrm{dis}(x,\partial G) \to 0} p_i(x) = +\infty$, uniformly, provided $x \in \mathrm{Int}(G)$.

(d) Let x(t) be a continuous n-vector valued function in class $C_*^1\,[0,1]$ with $x(t) \in \mathrm{Int}(G)$ for $t \in [0,1)$ and $x(1) \in \partial G$. Then for each $i = 1,2,\ldots$, $\int_0^1 p_i(x(t))dt = +\infty$. Here $C_*^1[0,1]$ denotes the class of n-vector valued functions on [0,1] each of which is absolutely continuous on [0,1] and possesses a uniform bound on its derivative.

OPTIMIZATION WITH A PENALTY FUNCTION

It is the purpose of this document to study means whereby solutions to the constrained optimization problem may be approximated. It is thus necessary to begin by defining what is here meant by an approximation procedure.

Let the initial point x_o be fixed and a (possibly) time-varying target set T(t) be defined. Let x(t) and u(t) be defined on $[t_o, t_1] \subseteq I$ with x(t) the response to the control u(t) via equation (1), $x(t_o) = x_o$ and $x(t_1) \in T(t_1)$. Let $x_k(t)$, $u_k(t)$ be similarly defined on intervals $[t_{ok}, t_{1k}] \subseteq I$ with $x_k(t_{ok}) = x_o, x_k(t_{1k}) \in T(t_{1k})$.

<u>DEFINITION 3.</u> The pairs of functions $(u_k(t), x_k(t))$ will be said to approximate the pair (u(t), x(t)) in case:

(a) $\lim_{k \to \infty} t_{ok} = t_o$, $\lim_{k \to \infty} t_{1k} = t_1$

(b) For each $t \in (t_o, t_1)$, $\lim_{k \to \infty} x_k(t) = x(t)$. (Note that $x_k(t)$ is well defined if k is sufficiently large.)

(c) Defining $u(t)$ and $u_k(t)$ to be zero at points $t \in I$ where they are previously undefined, $\lim_{k\to\infty} u_k = u$ in the weak topology of $L^2(I)$.

An approximation procedure for $(u(t),x(t))$ is any procedure whereby a sequence $\{(u_k(t), x_k(t))\}$ satisfying the above requirements may be obtained. It will be seen below the penalty function methods provide just such a procedure.

The following lemma is essential to the method to be described.

<u>LEMMA 1.</u> Let the optimization problem be defined as above. For each natural number k let $u_k \in \Delta$ and let x_k be the response to u_k via equation (1), both functions being defined on an interval $[t_{ok}, t_{1k}] \subseteq I$, where $x_k(t_{ok}) = x_o$ and $x_k(t_{1k}) \in T(t_{1k})$ is the first intersection of the trajectory x_k with the target $T(t)$. Assume that there is a positive number B such that $\|x_k(t)\| \leq B$, $t \in [t_{ok}, t_{1k}]$, for each k. Then there is a subsequence of the pairs u_k, x_k, which we continue to call $\{u_k, x_k\}$, with the properties

(a) $\lim_{k\to\infty} t_{ok} = t_o$, $\lim_{k\to\infty} t_{1k} = t_1$. $t_o, t_1 \in I$ and $t_o < t_1$.

(b) u_k converges to a function $u \in \Delta$, the convergence being in the weak topology of $L^2(I)$.

(c) Letting $x(t)$ denote the response to $u(t)$ via equation (1), $\lim_{k\to\infty} x_k(t) = x(t)$ for each $t \in (t_o,t_1)$, where $[t_o,t_1]$ is the domain of definition of x and u, as already indicated. This convergence is uniform in every closed subset of (t_o,t_1).

(d) $\lim_{k\to\infty} C(u_k) = C(u)$.

PROOF: The proof of this lemma is due to Lee and Markus (reference 1), except for the stated uniform convergence. Since the x_k and x obey (1) and are thereby absolutely continuous, and since $\|x_k\|$ is uniformly bounded by hypothesis, it may be concluded that $\dot{x}_k$ and $\dot{x}$ are uniformly bounded. Hence the stated uniform convergence follows from Lemma 2 which follows immediately.

LEMMA 2. Let $\{f_n(t)\}$ be a sequence of absolutely continuous functions converging pointwise on the compact interval I to the absolutely continuous function $f(t)$. If there exists a positive number B such that

(a) $|f'(t)| \leq B$ a.e.,

(b) $|f'_n(t)| \leq B$ a.e., for each n,

then $\{f_n(t)\}$ converges uniformly to $f(t)$ on I.

PROOF: Let $\delta > 0$ be chosen. Divide I into subintervals $I_1, I_2 \ldots, I_m$ of equal length such that $m \geq \frac{4B|I|}{\delta}$ where $|I|$ denotes the length of I. Let a point t_i be selected in each subinterval I_i, $i=1,2,\ldots,m$. Then n_o may be chosen so large that for $n \geq n_o$ $|f(t_i) - f_n(t_i)| < \frac{1}{2}\delta$, $i=1,2,\ldots,m$. Choose any $t \in I$. Suppose $t \in I_{i_o}$. Then let $n \geq n_o$.

$$|f(t) - f_n(t)| = |f(t_{i_o}) - f_n(t_{i_o}) + \int_{t_{i_o}}^{t} (f'(s) - f'_n(s))ds| \leq$$

$$\leq |f(t_{i_o}) - f_n(t_{i_o})| + \int_{t_{i_o}}^{t} (|f'(s)| + |f'_n(s)|)ds \leq \frac{1}{2}\delta +$$

$$+ \frac{\delta}{4B}\cdot 2B = \frac{1}{2}\delta + \frac{1}{2}\delta = \delta.$$

Hence $\{f_n(t)\}$ converges uniformly to $f(t)$.

It is desired to approximate solutions to the constrained optimization problem To this end a new problem is considered.

It is identical to the old unconstrained problem with the single exception that the cost function C(u) is replaced by

$$C_p(u) = \int_{t_o}^{t_1} [p(x(t)) + \sum_{j=1}^{m} h_j^o(t,x(t))u^j(t) + g^o(t,x(t))]dt \quad (3)$$

where p is a penalty function for the constraint set G.

THEOREM 1. A new unconstrained optimization problem with cost functional (3) is considered. p is a penalty function of the first kind for G, i.e., a member of a sequence of penalty functions of the first kind. This problem possesses an optimal solution u, x.

PROOF: A proof is presented in reference 1.

An analogous theorem for penalty functions of the second kind is now given.

THEOREM 2. A new unconstrained optimization problem with cost functional (3) is considered. p is a penalty function of the second kind for G. It is further assumed that the bounding hypothesis of Lemma 1 is satisfied. Then there is a solution $\overline{u}$, $\overline{x}$ to this optimization problem such that $\overline{x}(t)$ lies entirely in Int(G) on $[t_o,t_1] \subseteq I$ provided there is any $u^* \in \Delta$ whose response x^* is known to lie entirely in Int(G) on its domain of definition $[t_o^*, t_1^*] \subseteq I$.

PROOF: There exists a real number M such that $C_p(u) \geq M$ for all $u \in \Delta$ because of the boundedness of $g^o(t,x)$, $h_j^o(t,x)$, u(t) and the fact that p(x) is non-negative. Because of the hypothesized existence of $u^*(t)$ it may be assumed that M is the largest such number. If Δ contains only finitely many functions u(t) there is a function $\overline{u}(t) \in \Delta$ such that $C_p(\overline{u}) = M$. Suppose the response $\overline{x}(t)$

were such that it met ∂G. Then there would be in $(\bar{t}_o,\bar{t}_1)$ a smallest time t (because $\dot{\bar{x}}$ is bounded) such that $\bar{x}(t) \in \partial G$. But then using the condition (d) imposed upon penalty functions of the second kind we see that $\int_{t_o}^{t_1} p(x(t))dt = +\infty$. But then it cannot be that $C_p(\bar{u}) = M$. Hence $\bar{x}(t) \in \mathrm{Int}(G)$ for $t \in [\bar{t}_o,\bar{t}_1]$.

It is supposed that Δ contains infinitely many functions u(t). A sequence $\{u_k(t)\}$ is selected such that

$$\lim_{k\to\infty} C_p(u_k) = M. \tag{4}$$

By Lemma 1, a subsequence $\{u_{k_\ell}(t)\}$ of $\{u_k(t)\}$ converges weakly to a function $\bar{u}(t) \in \Delta$ and the responses $x_{k_\ell}(t)$ converge pointwise to $\bar{x}(t)$, uniformly in any compact subinterval of $(\bar{t}_o,\bar{t}_1)$, where $\bar{x}(t)$ is the response to $\bar{u}(t)$. Moreover defining C(u) by (2), $\lim_{\ell\to\infty} C(u_{k_\ell}) = C(\bar{u})$. Thus it remains only to show that

$$\lim_{\ell\to\infty} \int_{t_{ok_\ell}}^{t_{1k_\ell}} p(x_{k_\ell}(t))dt = \int_{\bar{t}_o}^{\bar{t}_1} p(\bar{x}(t))dt. \tag{5}$$

First of all, it is clear from (4) that the sequence

$$\left\{\int_{t_{ok_\ell}}^{t_{1k_\ell}} p(x_{k_\ell}(t))dt\right\}$$

is bounded. It is now claimed that there exists a compact subset $D \subset \mathrm{Int}(G)$ such that for each $= 1,2,3,\ldots$ and each $t \in [t_{ok_\ell},t_{1k_\ell}]$, $x_{k_\ell}(t) \in D$.

It is supposed for contradiction that this were not the case. Then there is a sequence of times t_ℓ such that dis $(x_{k_\ell}(t_\ell),\partial G) \to 0$.

Since the t_ℓ all lie within the compact interval I, a subsequence of them, which is still called $\{t_\ell\}$, converges to t^*. Since the $x_{k_\ell}(t_\ell)$ are uniformly bounded in norm by B, a subsequence, still called $x_{k_\ell}(t_\ell)$, converges to a point $x^* \in \partial G$. Because the derivatives of the x_{k_ℓ} are uniformly bounded, the sequences $\{t_\ell - t_{ok_\ell}\}$ and $\{t_{1k_\ell} - t_\ell\}$ are uniformly bounded away from zero. Thus a natural number ℓ_o and a compact interval $[\tau_1, \tau_2]$ can be found such that for $\ell \geq \ell_o$, $t_\ell \in [\tau_1, \tau_2]$ and $[\tau_1, \tau_2] \subset (t_{ok_\ell}, t_{1k_\ell})$ and also $[\tau_1, \tau_2] \subset (\bar{t}_o, \bar{t}_1)$. Now since the derivatives of the x_{k_ℓ} are uniformly bounded, $\lim_{\ell \to \infty} \{x_{k_\ell}(t_\ell) - x_{k_\ell}(t^*)\} = 0$. On the other hand, $\lim_{\ell \to \infty} x_{k_\ell}(t^*) = \bar{x}(t^*)$. Hence $\bar{x}(t^*) = x^* \in \partial G$. Then by condition (d) on penalty functions of the second kind, $\int_{\tau_1}^{t^*} p(\bar{x}(t))dt = +\infty$ and hence, since p is nonnegative

$$\int_{\tau_1}^{\tau_2} p(\bar{x}(t))dt = +\infty \tag{6}$$

(Set $p(\bar{x}(t)) = +\infty$ if $\bar{x}(t) \in \partial G$). Now for all $\ell \geq \ell_o$ the integral $\int_{\tau_1}^{\tau_2} p(x_{k_\ell}(t))dt$ is defined and there is a fixed real number $P > 0$ such that for all such ℓ

$$\int_{\tau_1}^{\tau_2} p(x_{k_\ell}(t))dt \leq P. \tag{7}$$

Moreover, for each $t \in [\tau_1, \tau_2]$, $\lim_{\ell \to \infty} x_{k_\ell}(t) = \bar{x}(t)$. Hence, remembering that $p(\bar{x}(t))$ may be $+\infty$, for each $t \in [\tau_1, \tau_2]$, $\lim_{\ell \to \infty} p(x_k(t)) = p(\bar{x}(t))$ in view of conditions (a) and (c) on penalty functions of the second kind. Hence, Fatou's Lemma may be applied with inequality (7) to see that $\int_{\tau_1}^{\tau_2} p(\bar{x}(t))dt$ exists and is $\leq$ P. But this is in contradiction with (6). Therefore there is a compact set $D \subset \mathrm{Int}\,(G)$ such that $x_{k_\ell}(t)$ always lies in D.

But p(x) is bounded for $x \in D$. Applying Lebesgue s dominated convergence theorem it is seen that

$$\begin{aligned} \lim_{\ell \to \infty} \int_I p(x_{k_\ell}(t))\, \psi_\ell(t)dt &= \int_I \lim_{\ell \to \infty} [p(x_{k_\ell}(t))\psi_\ell(t)]dt \\ &= \int_I p(\bar{x}(t))\, \bar{\psi}(t)dt \end{aligned} \tag{8}$$

where $\psi_\ell(t)$ is the characteristic function of $[t_{ok_\ell}, t_{1k_\ell}]$ and $\bar{\psi}(t)$ is the characteristic function of $[\bar{t}_o, \bar{t}_1]$. But then (8) proves (5) correct. Thus, it has been shown that $\lim_{\ell \to \infty} C_p(u_{k_\ell}) = C_p(\bar{u})$ and this completes the proof of Theorem 2.

OPTIMIZATION WITH SEQUENCES OF PENALTY FUNCTIONS

It has been shown that the optimization problem which includes a penalty function in its cost functional integrand has a solution. In what follows, sequences of penalty functions, as defined earlier, are used. It is assumed that the optimization problem is solved for each such function and now the study proceeds to study the sequence of solutions thus obtained. Theorem 3 establishes the existence of exact solutions to the constrained phase coordinate optimization problem as limits of sequences of such solutions provided (u,x) is approximable (Definition 4) in the interior of G. Theorem 4 establishes that if the system is linear, normal, and autonomous and if there is one control that maintains the system trajectories within the interior of a convex constraint set, then (u,x) is approximable from the interior. Theorem 5 proves the existence of a solution to the general bounded phase coordinate control optimization problem by means of sequences of penalty functions of the first kind. In addition, a discussion of the requirement on approximability from the interior is given.

Consider therefore sequences $\{p_i(x)\}$ of penalty functions. For a control $u(t)$ $Cp_i(u)$ is defined by

$$Cp_i(u) = \int_{t_o}^{t_1} [p_i(x(t)) + g^o(t,x(t)) + \sum_{j=1}^{m} h_j^o(t,x(t))u^j(t)]dt. \tag{9}$$

For each i let $u_i(t)$ with response $x_i(t)$, $x_i(t_{oi}) = x_o$, $x_i(t_{1i}) \in T(t_{1i})$ be a solution to the unconstrained optimization problem with cost functional $Cp_i(u)$.

<u>DEFINITION 4.</u> Let $u(t)$ with response $x(t)$, $x(t_o) = x_o$, $x(t_1) \in T(t_1)$, belong to Δ. The pair (u,x) is said to be approximable in the interior of G on I if there exists a sequence of intervals $[t_o + \delta_o^k \;\; t_1 + \delta_1^k] \subseteq I$ with $\lim_{k\to\infty} \delta_o^k = \lim_{k\to\infty} \delta_1^k = 0$ and a sequence of pairs (u_k,x_k), $u_k \in \Delta$, $x_k(t)$ the response to $u_k(t)$, $x_k(t_o + \delta_o^k) = x_o$, $x_k(t_1 + \delta_1^k) \in T(t_1 + \delta_1^k)$, such that

(a) $\{u_k(t)\}$ converges weakly to $u(t)$,

(b) $x_k(t)$ converges pointwise to $x(t)$ and uniformly on any compact subinterval of (t_o,t_1) and $x_k(t) \in \text{Int}\ (G)$ for $t \in [t_o + \delta_o^k, t_1 + \delta_1^k]$.

<u>THEOREM 3.</u> Let $\{p_i(x)\}$ be a sequence of penalty functions of either first or second kind for G. Let $u_i(t)$, $x_i(t)$ be a solution of the optimization problem with cost functional $Cp_i(u)$ for each i as described above. Assume $\|x_i(t)\| \leq B$ for each i, t. If for each $u(t) \in \Delta$ the corresponding pair (u,x) is approximable in the interior of the

interior of G on I, then there exists a subsequence $\{u_{i_k}(t)\}$ converging weakly to $\bar{u}(t) \in \Delta$ with the responses $x_{i_k}(t)$ converging pointwise to $\bar{x}(t)$, uniformly on compact subintervals of $(\bar{t}_o, \bar{t}_1)$, and $\bar{u}(t)$ and $\bar{x}(t)$ are a solution of the constrained optimization problem with cost functional $C(u)$.

PROOF: Everything has been proved in Lemma 1 except the statement that $\bar{u}(t)$ and $\bar{x}(t)$ are a solution of the constrained optimization problem with cost functional $C(u)$.

For contradiction, it is supposed there is a control $u(t) \in \Delta$ such that

$$C(u) = C(\bar{u}) - c \tag{10}$$

where c is a positive real number. Consider a sequence of pairs $(\hat{u}_\ell, \hat{x}_\ell)$ which approximates the pair (u,x) in the interior of G. Assume that $u(t)$ is defined on the interval $[t_o, t_1]$ and that $\hat{u}_\ell(t)$ is defined on $[t_o + \delta_o^\ell, t_1 + \delta_1^\ell]$.

Now

$$|C(u) - Cp_{i_k}(\hat{u}_\ell)| =$$

$$\Big|\int_{t_o}^{t_1} \{g^o(t,x(t)) + \sum_{j=1}^{m} h_j^o(t,x(t))u^j(t)\}dt$$

$$- \int_{t_o+\delta_o^\ell}^{t_1+\delta_o^\ell} \{p_{i_k}(\hat{x}_\ell(t)) + g^o(t,\hat{x}_\ell(t)) + \sum_{j=1}^{m} h_j^o(t,\hat{x}_\ell(t))\hat{u}_\ell^j(t)\}dt\Big|$$

$$\le \int_{t_o+\delta_o^\ell}^{t_1+\delta_o^\ell} |p_{i_k}(\hat{x}_\ell(t))|dt + M(|\delta_o^\ell| + |\delta_1^\ell|) \tag{11}$$

$$+ \int_{J_\ell} \{|g^o(t,\hat{x}(t)) - g^o(t,\hat{x}_\ell(t))|$$

$$+ \Big|\sum_{j=1}^{m} (h_j^o(t,x(t))\, u^j(t) - h_j^o(t,\hat{x}_\ell(t))\, \hat{u}_\ell^j(t))\Big|\}dt$$

where M is a bound for the function $g^{o}(t,x) + \sum_{j=1}^{m} h_j^{o}(t,x)u^j$ valid for $t \in I$, $x \in G$, $u \in \Omega$, and $J_\ell = [t_o,t_1] \cap [t_o + \delta_o^\ell, t_1 + \delta_1^\ell]$.

Let $\epsilon = c/\mu$, where μ is a positive integer which will be fixed later. Let ℓ_o be chosen so that $\ell \geq \ell_o \Rightarrow$

$$M(|\delta_o^\ell| + |\delta_1^\ell|) < \epsilon. \tag{12}$$

Since $\hat{x}_\ell(t)$ converges uniformly to $x(t)$ on any compact subinterval of $[t_o,t_1]$ and g^o is continuous, and hence uniformly continuous on $\mathrm{Int}(G \cap \{x \mid \|x\| \leq B\})$, ℓ_1 can clearly be chosen so that $\ell \geq \ell_1 \Rightarrow$

$$\int_{J_\ell} |g^o(t,x(t)) - g^o(t,\hat{x}_\ell(t))| dt < \epsilon \tag{13}$$

Now

$$\int_{J_\ell} \left\{ \sum_{j=1}^{m} (h_j^o(t,x(t))u^j(t) - h_j^o(t,\hat{x}_\ell(t)\hat{u}_\ell^j(t) | \right\} dt \tag{14}$$

$$\leq \int_{J_\ell} \left\{ \left| \sum_{j=1}^{m} (h_j^o(t,x(t))(u^j(t) - \hat{u}_\ell^j(t))) \right| \right\} dt$$

$$+ \int_{J_\ell} \left\{ \sum_{j=1}^{m} (h_j^o(t,x(t)) - h_j^o(t,\hat{x}_\ell(t)))\hat{u}_\ell^j(t) | \right\} dt$$

Again using the uniform continuity of h_j^o together with the uniform convergence of the $\hat{x}_\ell(t)$ to $x(t)$ on compact subintervals of $[t_o,t_1]$ and the boundedness of $u(t)$, ℓ_2 can be found so that $\ell \geq \ell_2 \Rightarrow$

$$\int_{J_\ell} \left\{ \left| \sum_{j=1}^{m} (h_j^o(t,x(t)) - h_j^o(t,\hat{x}_\ell(t)))\hat{u}_\ell^j(t) \right| \right\} dt < \epsilon. \tag{15}$$

Since $\hat{u}_\ell(t)$ converges weakly to $u(t)$ ℓ_3 can be found so that $\ell \geq \ell_3 \Rightarrow$

$$\int_{J_\ell} \left\{ \left| \sum_{j=1}^{m} (h_j^o(t,x(t))(u^j(t) - \hat{u}_\ell^j(t))) \right| \right\} dt < \epsilon. \tag{16}$$

Let $\hat{\ell} = \max \{\ell_o, \ell_1, \ell_2, \ell_3\}$. Then for $\ell \geq \hat{\ell}$

$$|C(u) - C_{p_{i_k}}(\hat{u}_\ell)| \leq \int_{t_o+\delta_o^\ell}^{t_1+\delta_1^\ell} |p_{i_k}(\hat{x}_\ell(t))| dt + 4\epsilon. \tag{17}$$

Since $\{\hat{x}_\ell(t) | t \epsilon [t_o + \delta_o^{\hat{\ell}}, t_1 + \delta_1^{\hat{\ell}}]\}$ is a compact subset of Int (G), K_o may be chosen such that $k \geq K_o \Rightarrow |p_{i_k}(\hat{x}_\ell(t))| < \frac{\epsilon}{(t_1+\delta_1^\ell - t_o - \delta_o^\ell)}$.

Then clearly

$$|C(u) - Cp_{i_k}(\hat{u}_\ell)| < 5\epsilon \text{ for such } k \tag{18}$$

Set $\mu = 10$ and

$$|C(u) - Cp_{i_k}(\hat{u}_\ell)| < c/2 \text{ for such } k. \tag{19}$$

Now clearly for each k

$$Cp_{i_k}(u_{i_k}) \geq C(u_{i_k}). \tag{20}$$

As proved in Theorem 1,

$$\lim_{k \to \infty} C(u_{i_k}) = C(\overline{u}). \quad \text{Hence} \tag{21}$$

$$\underline{\lim}_{k \to \infty} Cp_{i_k}(u_{i_k}) \geq C(\overline{u}). \tag{22}$$

Combining (10), (19), (22) it is clear that $k \geq k_o$ can be chosen so large that

$$Cp_{i_k}(u_{i_k}) > Cp_{i_k}(u\ ). \tag{23}$$

But this contradicts the choice of the control u_{i_k}. Hence (10) is impossible and the control $\overline{u}(t)$ with its response $\overline{x}(t)$ is indeed a solution of the constrained optimization problem. Q.E.D.

The necessity of the condition of approximability from the interior of G for penalty functions of the second kind is now shown by an example. Then it will be shown that there is a class of problems in which this condition is fulfilled.

Consider the linear autonomous system

$$\dot{x}_1 = x_2 \qquad |u| \leq 1$$
$$\dot{x}_2 = u \tag{24}$$

When the Maximum Principle of Pontrjagin is applied it is seen that the time optimal paths are segments of the parabolas

$$x_1 = \pm \frac{(x_2)^2}{2} + C$$

as shown in Fig. 1.

Let G be the shaded region and take $x_o \in \text{Int}\,(G)$ as shown in Fig. 2. T = 0, the origin, is the fixed target set. The optimal trajectory from x_o to T is the arc of the parabola $x_1 = \frac{-(x_2)^2}{2}$ between x_o and 0. It is easily verified that every solution of (24) for $|u| \leq 1$ which originates at x_o lies on this parabola or in the region $x_1 > \frac{-(x_2)^2}{2}$ as long as it does not cross the line $x_2 = 0$. Since a segment of the parabola $x_1 = \frac{-(x_2)^2}{2}$ forms part of the boundary of G it is clear that the optimal path from x_o to 0 cannot be approximated from the interior of G. Hence if penalty functions of the second kind were used to constrain solutions to G the attempt would end in failure. Note that, as sketched, there are trajectories from x_o to 0 lying in Int (G) but no sequence of such trajectories converges to the optimal trajectory.

The following Lemma is a familiar result from reference 1.

LEMMA 3: Consider the autonomous system

$$\dot{x}^i = f^i(x^1,\ldots x^n, u^1,\ldots u^m), \; i = 1,2,\ldots,n \qquad (S)$$

where $f(x,u)$ is in C^1 in R^{n+m}. We consider measurable controls $u(t)$ lying in a compact, convex subset Ω of R^m such that Ω contains the origin in its interior. Let $A = \frac{\partial f}{\partial x}(0,0)$, $B = \frac{\partial f}{\partial u}(0,0)$ and assume

(a) $f(0,0) = 0$.

(b) There exists a vector $v \in R^m$ such that Bv lies in no proper invariant subspace of A. (This is equivalent to saying that Bv, ABV, $A^2Bv,\ldots,A^{n-1}Bv$ are linearly independent.)

Then for every small negative number T there exists a neighborhood N_T of $x=0$ such that if $x_o \in N_T$, a control $u(t)$ may be found such that if $x(t)$ obeys (S) with $u = u(t)$ and $x(T) = x_o$, then $x(0) = 0$.

This lemma leads to a theorem giving sufficient conditions for approximability from the interior for a certain class of linear autonomous control systems.

DEFINITION 4. Consider the linear autonomous control system

$$\dot{x} = Ax + Bu \qquad (25)$$

The system (25) is called normal if there exists a vector $v \in R^m$ such that $Bv, ABv,\ldots, A^{n-1}Bv$ are linearly independent.

THEOREM 4. Let G be a closed convex subset of R^n and let x_o and the origin 0 lie in the interior of G. Let $u(t)$ defined on $[t_o,t_1]$ be such that the response $x(t)$ with $x(t_o) = x_o$ has $x(t_1) = 0$ and $x(t) \in G$ for $t \in [t_o,t_1]$. If there is any control

$\overline{u}(t)$ defined on $[t_o,t_1]$ such that the response $\overline{x}(t)$ with $\overline{x}(t_o) = x_o$ lies in Int (G) for $t \in [t_o,t_1]$ then the pair (u,x) is approximable from the interior of G if (1) is a normal linear system.

PROOF: For each natural number k let $u_k(t) = (1-\frac{1}{k})u(t)+\frac{1}{k}\overline{u}(t)$, $t \in [t_o,t_1]$. Let $x_k(t)$ be the response to $u_k(t)$ with $x_k(t_o) = x_o$. Now by the variation of parameters formula

$$x(t) = e^{A(t-t_o)}x_o + e^{A(t-t_o)}\int_{t_o}^{t} e^{-A(s-t_o)}Bu(s)ds \tag{26}$$

$$\overline{x}(t) = e^{A(t-t_o)}x_o + e^{A(t-t_o)}\int_{t_o}^{t} e^{-A(s-t_o)}B\overline{u}(s)ds$$

and hence

$$x_k(t) = e^{A(t-t_o)}x_o + e^{A(t-t_o)}\int_{t_o}^{t} e^{-A(s-t_o)}B[(1-\tfrac{1}{k})u(s)+\tfrac{1}{k}\overline{u}(s)]ds \tag{27}$$

$$= (1-\tfrac{1}{k})\,x(t) + \tfrac{1}{k}\,\overline{x}(t).$$

Since $\overline{x}(t) \in$ Int (G), $x(t) \in$ Int (G) and G is convex, $x_k(t) \in$ Int (G) for each k and each t. Now

$$\lim_{k\to\infty} x_k(t_1) = 0. \tag{28}$$

Using the fact that $\dot{x}_k$ must be bounded together with Lemma 3 it can be stated: For each sufficiently large k there exists a non-negative number δ_k such that $u_k(t)$ may be defined on $[t_o,t_1+\delta_k]$ in such a manner that $x_k(t_1+\delta_k) = 0$. $x_k(t) \in$ Int (G) for $t \in [t_o,t_1+\delta_k]$, and $\|x_k(t)\| < \epsilon_k$ for $t \in [t_1, t_1+\delta_k]$. The δ_k may be chosen so that $\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \epsilon_k = 0$. Since x(t) is uniformly continuous on $[t_o,t_1]$ and $\lim_{k\to\infty} x_k(t) = x(t)$ uniformly there and since $u_k(t)$ and

$u(t)$ are uniformly bounded on $[t_1, t_1 + \delta_k]$ for all k and $\lim_{k \to \infty} u_k(t) = u(t)$ uniformly on $[t_o, t_1]$ it is seen that the sequence (u_k, x_k) defined on the sequence of intervals $[t_o, t_1 + \delta_k]$ approximates (u,x) from the interior of G and this completes the proof.

The author is confident that much more general theorems concerning approximability from the interior are actually true. The proof of this assertion appears at this time to be quite difficult.

Penalty functions of the first kind which satisfy the additional requirement:

$$\lim_{i \to \infty} p_i(x) = 0, \text{ uniformly for } x \in G \tag{29}$$

are now considered. A special instance of this situation occurs when G is described by

$$G: \{x \mid p(x) = 0\} \tag{30}$$

where $p(x)$ is a continuous, non-negative function which is > 0 for $x \notin G$. Then one can take $p_i(x) = ip(x)$, thereby obtaining a sequence of penalty functions of the first kind which satisfy (31). This approach to the problem of minimizing functionals subject to constraints is originally due to Courant. (See Refs. 3, 4). In the supplementary notes to Courant's "Calculus of Variations" (Ref. 3) the following theorem is proved by Martin Kruskal and Hanan Rubin:

<u>THEOREM 5</u> If

(a) $\Phi(p)$ and $\psi(p)$ are lower semi-continuous real valued functions on a convergence space S (i.e., any space in which a notation of convergence is defined);

(b) $\psi(p) \geq 0$ for all p in S and there exist points in S for which $\psi(p) = 0$;

(c) A denotes the problem: Find a point satisfying the side condition

$$\psi(p) = 0$$

at which $\Phi(p)$ takes on its least value for all p in S satisfying the side condition;

(d) A_t denotes the problem: Find a point for which $\Phi(p) + t\psi(p)$ takes on its least value for all p in S; and

(e) There exists a sequence $\{t_n\}$ of positive real numbers, a sequence $\{p_n\}$ of points in S, and a point p_∞ in S such that $t_n \to \infty$ as $n \to \infty$, p_n solves A_{t_n} and $p_n \to \infty$ as $n \to \infty$; then: p_∞ solves A.

The proof of this theorem is not presented here. In order to apply it to the present problem the following identifications are made: S consists of all pairs of functions $(u(t), x(t))$ defined on I, $u(t) \in \Omega$ such that $x(t)$ is the response to $u(t)$ on I and there are times $t_o, t_1 \in I$ with $x(t_o) = x_o$, $x(t_1) \in T(t_1)$, $\|x(t)\| \leq B$ uniformly. S is taken as a subspace of $L^2(I) \times C(I)$, the topology in $L^2(I)$ being the weak topology, and in $C(I)$ the topology of the sup norm. Then the product topology defines the convergence in S. $\Phi(u(t), x(t))$ is defined to be $C(u)$, the integral used in $C(u)$ being taken between t_o, t_1, t_1 being the first time $x(t)$ meets $T(t)$ after time t_o. It is clear that $\Phi(u(t), x(t))$ is continuous with respect to the chosen topology of S.

$$\psi(u(t), x(t)) \text{ is defined to be } \int_{t_o}^{t_1} (p(x(t))dt \geq 0.$$

Again ψ is continuous on S since it depends only on x(t) and is continuous in C(I).

The condition that there are points in S for which $\psi(u(t), x(t)) = 0$ is precisely the condition that there is a $u(t) \in \Delta$ such that the response x(t) lies in G for $t_o \leq t \leq t_1$. Then Lemma 2 may be used to obtain condition (e) and the following theorem may be stated as a consequence of Theorem 5.

THEOREM 6. Let p(x) satisfy (30) and $p_i(x) = ip(x)$, p(x) as defined above. Then Theorem 3 holds for this sequence of penalty functions without the requirement on approximability from the interior.

The same result may be proved for sequences of penalty functions which satisfy equation (29). It is embodied in the following

THEOREM 7. Theorem 3 holds for a sequence of penalty functions of the first kind which satisfy (29) without the condition on approximability from the interior.

PROOF: The proof is the same as that in Theorem 3 with the functions (u_ℓ, x_ℓ) being replaced by (u, x).

It is clear that Theorem 7 implies Theorem 6 but Theorem 6 is included to show the usefulness of the Kruskal-Rubin approximation theorem, i.e., Theorem 5.

CONCLUSION

It has been shown that with suitable hypotheses optimal constrained phase trajectories may be approximated to any desired degree of accuracy by unconstrained solutions of problems wherein the cost has been augmented by a penalty function.

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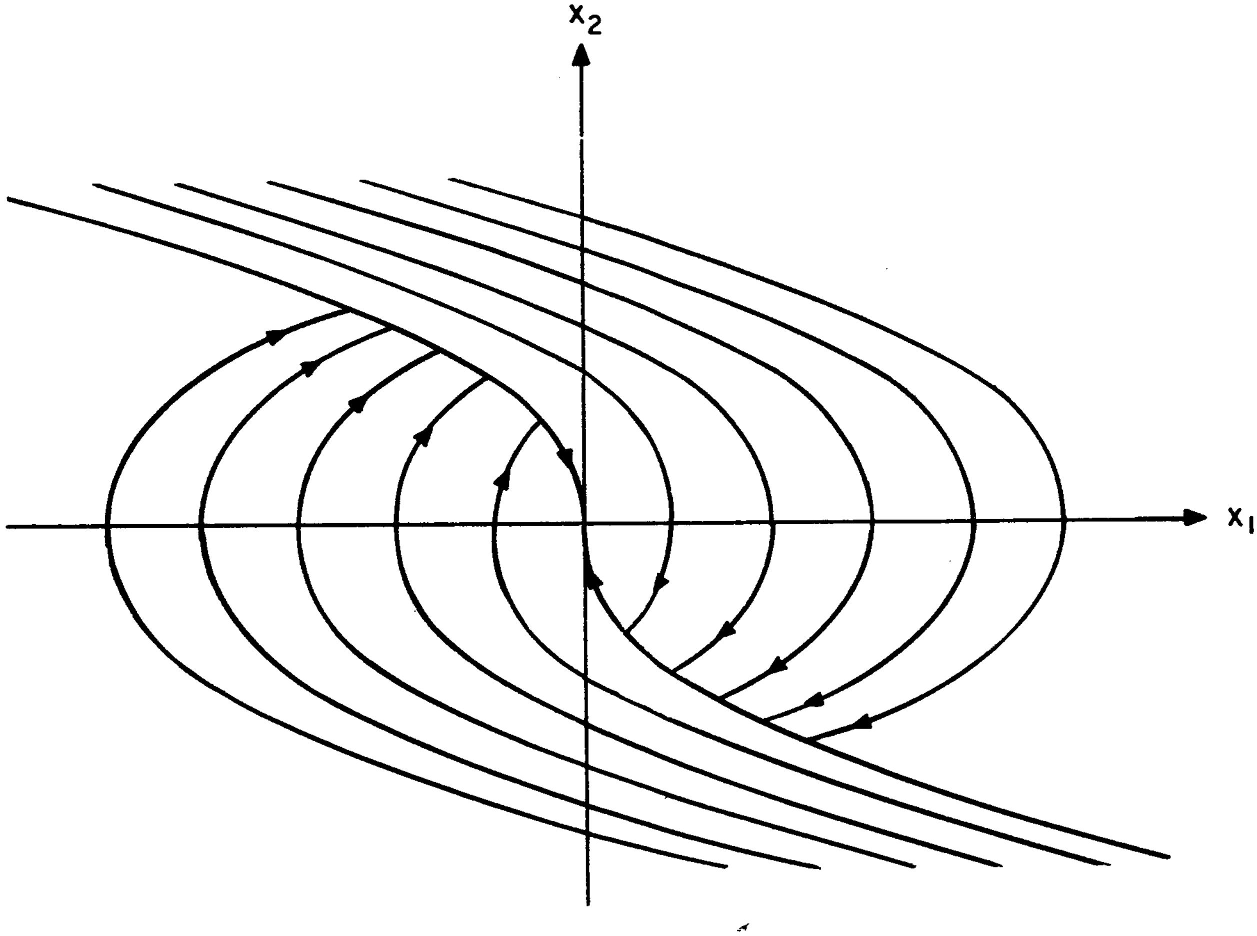

Figure 1. Time-Optimal Trajectories for $\ddot{x} = u$

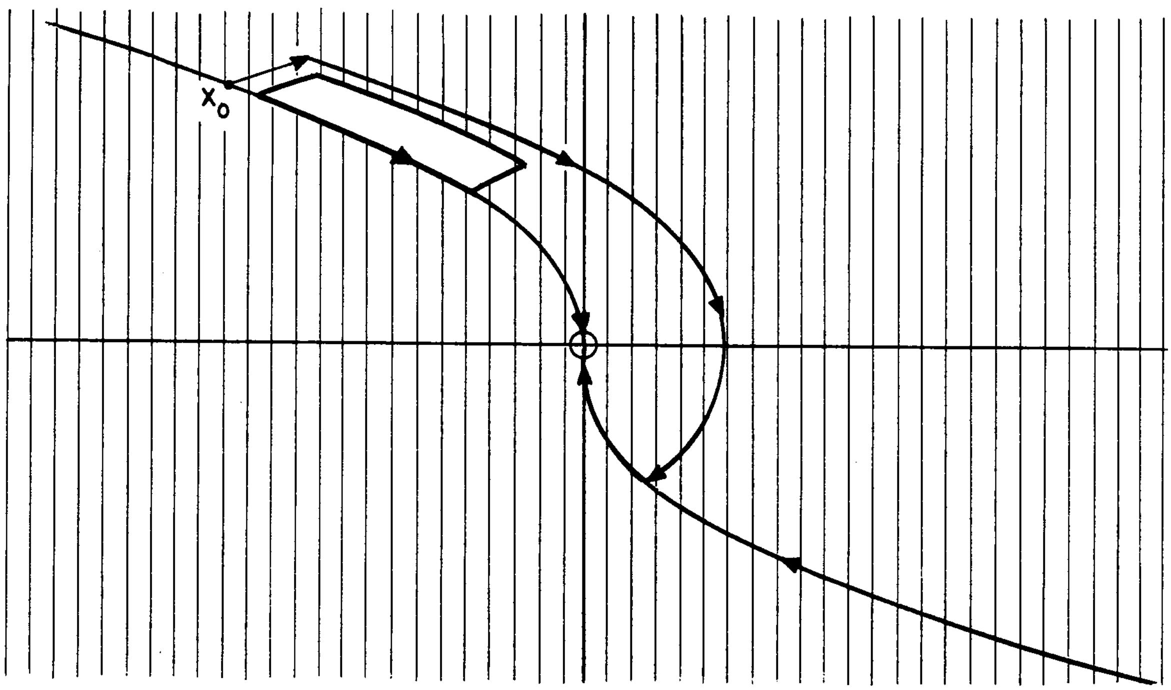

Figure 2. Not Approximable from the Interior